Rings over which all (finitely generated strongly) Gorenstein projective modules are projective

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Abstract. The main aim of this paper is to investigate rings over which all (finitely generated strongly) Gorenstein projective modules are projective. We consider this propriety under change of rings, and give various examples of rings with and without this propriety.

Key Words. (Strongly) Gorenstein projective and flat modules, Gorenstein global dimension, weak Gorenstein global dimension, trivial ring extensions, subring retract, (n, d)-ring, n-Von Neumann regular ring.

1 Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital. For an R-module M, we use $\mathrm{pd}_R(M)$ to denote the usual projective dimension of M. $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ are, respectively, the classical global and weak global dimensions of R. It is convenient to use "m-local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal m.

In 1967-69, Auslander and Bridger [1, 2] introduced the G-dimension for finitely generated modules over Noetherian rings. Several decades later, this homological dimension was extended, by Enochs and Jenda [9, 10], to the Gorenstein projective dimension of modules that are not necessarily finitely generated and over non-necessarily Noetherian rings. And, dually, they defined the Gorenstein injective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [12] introduced the Gorenstein flat dimension.

In the last years, the Gorenstein homological dimensions have become a vigorously active area of research (see [7, 11] for more details). In 2004, Holm [20] generalized several results which already obtained over Noetherian rings. Recently, in [5] the authors introduced particular cases of

Gorenstein projective, injective, and flat modules, which are called respectively, strongly Gorenstein projective, injective and flat modules, which are defined, respectively, as follows:

Definitions 1.1 ([5]) *1. A module M is said to be strongly Gorenstein projective, if there exists a complete projective resolution of the form*

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}(f)$.

2. The strongly Gorenstein injective modules are defined dually.

3. A module M is said to be strongly Gorenstein flat, if there exists a complete flat resolution of the form

$$\mathbf{F} = \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}(f)$.

The principal role of the strongly Gorenstein projective modules is to give a simple characterization of Gorenstein projective modules, as follows:

Theorem 1.2 ([5], Theorem 2.7) *A module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module.*

The important of this last result manifests in showing that the strongly Gorenstein projective modules have simpler characterizations than their Gorenstein correspondent modules. For instance:

Proposition 1.3 ([5], Proposition 2.9) *A module M is strongly Gorenstein projective if and only if there exists a short exact sequence of modules:*

$$0 \to M \to P \to M \to 0,$$

where P is projective, and $\mathrm{Ext}(M, Q) = 0$ for any projective module Q.

In order to give an answer to the question "when is a finitely generated torsionless module projective?" Luo and Huang proved, for a commutative Artinian ring R, that a Gorenstein projective R-module M is projective if $\mathrm{Ext}^i_R(M, M) = 0$ for any $i \geq 1$ [16, Theorem 4.7]. And over Noetherian local rings Takahashi proved that G-regular rings are the rings over which all $Gdim(M) = \mathrm{pd}(M)$ for any R-module M [22, Proposition 1.8]. Over a local ring (R, m) satisfies that $m^2 = 0$ and R is not a Gorenstein ring, Yoshino proved that every R-module of G-dimension zero is free [23, Proposition 2.4]. In this paper, we are concerned with a global question. Namely, we study the following two classes of rings: rings over which all Gorenstein projective modules are projective and rings over which all finitely generated strongly Gorenstein projective modules are projective. In Section 2, we show, that first class coincides with the class of rings over which all strongly Gorenstein projective modules are projective (see Theorem 2.1). Furthermore, in the same result, we show that a ring R belongs in this class if and only if $\mathrm{Ext}^1_R(M, M) = 0$ for any strongly Gorenstein projective R-module M if and only if $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ for any R-module M. After, we study the second class over which all finitely generated strongly Gorenstein projective modules are projective. Then, we study the transfer of this property in some extensions of rings. In section 3, we give some examples of rings with and without this property.

2 Rings over which all (finitely generated strongly) Gorenstein projective modules are projective

We start this section with investigating rings satisfy the property "all Gorenstein projective R-modules are projective". In the next Theorem, we see some conditions equivalent to this property.

Theorem 2.1 *Let R be a ring. The following conditions are equivalent:*

1. *All Gorenstein projective R-modules are projective;*
2. *All strongly Gorenstein projective R-modules are projective;*
3. *For any strongly Gorenstein projective R-module M, $\mathrm{Ext}^1_R(M,M)=0$;*
4. *For any R-module M, $\mathrm{Gpd}_R(M)=\mathrm{pd}_R(M)$.*

Proof. (2) ⇒ (3). Is obvious
(3) ⇒ (2). Let M be a strongly Gorenstein projective module, from Proposition 1.3, there is an exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0 \qquad (\star)$$

where P is projective. And since $\mathrm{Ext}^1_R(M,M)=0$, the sequence $(\star)$ spilt and M is a direct summand of P, then M is projective.
(2) ⇒ (1). Follows immediately from Theorem 1.2.
(1) ⇒ (2). Obvious (since all strongly Gorenstein projective modules are Gorenstein projective).
(4) ⇒ (1). Obvious.
(1) ⇒ (4). Let M be an R-module, it is known that $\mathrm{Gpd}_R(M)\leq \mathrm{pd}_R(M)$. Then it remains to prove that $\mathrm{pd}_R(M)\leq \mathrm{Gpd}_R(M)$. If $\mathrm{Gpd}_R(M)=\infty$ it is obvious. Let m be a positif integer and $\mathrm{Gpd}_R(M)=m<\infty$. From [20, Definition 2.8], M has a Gorenstein projective resolution of length m. Then, $\mathrm{pd}_R(M)\leq m=\mathrm{Gpd}_R(M)$ since all Gorenstein projective modules are projective. Therefore, $\mathrm{Gpd}_R(M)=\mathrm{pd}_R(M)$. ■

Throughout the remainder of this paper, we study rings satisfy each of the following conditions equivalent:

Theorem 2.2 *Let R be a ring. The following conditions are equivalent:*

1. *All finitely generated strongly Gorenstein projective R-modules are projective;*
2. *All finitely generated strongly Gorenstein projective R-modules are flat;*
3. *All finitely presented strongly Gorenstein flat R-modules are projective;*
4. *All finitely presented strongly Gorenstein flat R-modules are flat;*
5. *For any finitely generated strongly Gorenstein projective R-module M, $\mathrm{Ext}_R(M,M)=0$.*

Proof. (1) ⇔ (2) and (4) ⇒ (1). Follows immediately from [5, Proposition 3.9], and since every finitely presented flat R-module is projective.
(1) ⇒ (3). Follows from [5, Proposition 3.9].
(3) ⇒ (4). Obvious.
(1) ⟺ (5). Similar to the proof of Theorem 2.1. ■

Recall, for an extension of rings $A \subseteq B$, that A is called a module retract of B if there exists an A-module homomorphism $f : B \longrightarrow A$ such that $f_{/A} = \mathrm{id}_{/A}$. The homomorphism f is called a module retraction map. If such map f exists, B contains A as a direct summand A-module. In the next main result, we study the property "all finitely generated strongly Gorenstein projective modules are projective" in retract rings.

Theorem 2.3 *Let A be a retract subring of R, $(R = A \oplus_A E)$, such that E is a flat A-module. Then, if the property, all finitely generated strongly Gorenstein projective modules are projective, holds in R, then, it holds in A too.*

Proof. We prove first that $M \otimes_A R$ is a finitely generated strongly Gorenstein projective R-module, for any finitely generated strongly Gorenstein projective A-module M. From [5, Proposition 2.12], there exists an exact sequence of A-modules:

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is a finitely generated projective A-module. $P \otimes_A R$ is a finitely generated projective R-module, and since R is a flat A-module, it follows that the sequence of R-modules:

$$0 \longrightarrow M \otimes_A R \longrightarrow P \otimes_A R \longrightarrow M \otimes_A R \longrightarrow 0$$

is exact. It remains only to show that $\mathrm{Ext}_R(M \otimes_A R, R) = 0$. Therefore, $\mathrm{Ext}_A(M, R) = 0$ (since R is an A-module flat and from [5, Proposition 2.12]). On the other hand, from [6, Proposition 4.1.3], $\mathrm{Ext}_R(M \otimes_A R, R) \cong \mathrm{Ext}_A(M, R) = 0$. Then $M \otimes_A R$ is a finitely generated strongly Gorenstein projective R-module and by hypothesis it is projective. To complete the proof, we will show that $\mathrm{Ext}^k_A(M, N) = 0$ for any integer $k \geq 1$ and for any A-module N. It is known that $\mathrm{Ext}^k_A(M, N \otimes_A R) \cong \mathrm{Ext}^k_R(M \otimes_A R, N \otimes_A R) = 0$, (from [6, Proposition 4.1.3]). Namely, $\mathrm{Ext}^k_A(M, N)$ is a direct summand of $\mathrm{Ext}^k_A(M, N \otimes_A R)$, as A-modules (since A is a direct summand of R as A-module). Then, $\mathrm{Ext}^k_A(M, N) = 0$ and M is a projective A-module as desired. ■

Next we study the transfer of the property, all finitely generated strongly Gorenstein projective modules are projective, in polynomial rings.

Corollary 2.4 *Let R be a ring and X an indeterminate over R. If $R[X]$ satisfies the conditions equivalent of Theorem 2.2, then R satisfies it too.*

Proof. Note first that this Corollary is a particular case of Theorem 2.3 above, but here we get an other proof. Let M be a finitely generated strongly Gorenstein projective R-module. Since $\mathrm{pd}_R(R[X])$ is finite, and from [4, Theorem 2.11], $M[X]$ is a finitely generated strongly Gorenstein projective $R[X]$-module, so $M[X]$ is a projective $R[X]$-module. Then, [21, Lemma 9.27] gives that M is a projective R-module. ■

Recall, let A be a ring and let E an A-module. The trivial ring extension of A by E is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. We define similarly $J := I \propto E'$, where I is an ideal of A and E' is an A-submodule of E such that $IE \subseteq E'$. Then J is an ideal of R and, if J is a finitely generated ideal, then so is I [15, Theorem 25.1]. Trivial ring extensions have been studied extensively; the work is summarized in [13, 14, 15]. These extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance,[14, 15, 17, 18]. As a direct application of Theorem 2.3 above we have:

Corollary 2.5 *Let A be a ring and E a flat A-module. If the property, all finitely generated strongly Gorenstein projective modules are projective, holds in $R = A \propto E$, then it holds in A too.*

It is well-known that the structures of ideals are simple more then the structures of modules, and the study of any property under ideals gives a large class of examples and solving some problems. In the following result we study the transfer of the property "all finitely generated strongly Gorenstein projective ideals are projective" between an integral domain D and its trivial ring extension $D \propto K$ where $K = qf(D)$.

Theorem 2.6 *Let (D, m) be an integral domain m-local not field and $K = qf(D)$. Let $R = D \propto K$, then the following conditions are equivalent:*

1. *R satisfies all finitely generated strongly Gorenstein projective ideals are projective.*
2. *D satisfies all finitely generated strongly Gorenstein projective ideals are projective.*

To prove this theorem we need the following Lemma.

Lemma 2.7 *Let (D, m) be an integral domain m-local not field and $K = qf(D)$. Let $R = D \propto K$. If the property, all finitely generated strongly Gorenstein projective ideals are projective, holds in D, then $0 \propto I$ can not be a strongly Gorenstein projective ideal of R, for any finitely generated ideal I of D.*

Proof. Assume, on the contrary, that $0 \propto I$ is a finitely generated strongly Gorenstein projective ideal of R, for some finitely generated ideal I of D. We prove first that $0 \propto I$ is a strongly Gorenstein flat D-module. From [5, Proposition 2.12], there exists an exact sequence of R-modules:

$$0 \longrightarrow 0 \propto I \longrightarrow P \longrightarrow 0 \propto I \longrightarrow 0 \qquad (\star)$$

where P is finitely generated projective R-module, then P is a free R-module (since R is a local ring). Therefore, $(\star)$ is also an exact sequence of D-module and since R is a flat D-module, P is also a flat D-module. From [5, Proposition 3.6], it remains to prove that $Tor_D(E, 0 \propto I) = 0$, for any injective D-module E. Thus, $Tor_D(E, 0 \propto I) \cong Tor_R(\mathrm{Hom}_D(R, E), 0 \propto I) = 0$ (from [6, Proposition 4.1.1]). Then, $0 \propto I$ is a finitely generated strongly Gorenstein flat D-module and since D is an m-local domain $0 \propto I$ is a strongly Gorenstein projective D-module (from [5, Corollary 3.10]). On the other hand, $0 \propto I \cong I$ as D-module. It follows that I is a finitely generated strongly Gorenstein projective ideal of D and by hypothesis projective and since D is m-local I is free. Then, I is a principal ideal of D. Let $I = Da$ where $a \in I$. There exists an exact sequence of R-module:

$$0 \longrightarrow 0 \propto K \longrightarrow R \xrightarrow{u} 0 \propto I \longrightarrow 0 \qquad (\star\star)$$

where $u((b, e) = (0, ba)$. Therefore, from Schanuel's lemma applied to the sequences $(\star)$ and $(\star\star)$ we have $0 \propto K \oplus_R P = 0 \propto I \oplus_R R$. Hence, we conclude that $0 \propto K$ is a finitely generated ideal of R. Contradiction, since K is not a finitely generated D-module (since D is not a field). Then $0 \propto I$ can not be a strongly Gorenstein projective ideal of R as desired. ■

Proof. of Theorem 2.6. (1) ⇒ (2). Let I be an ideal of D finitely generated strongly Gorenstein projective. First we show that $I \otimes_D R = I \propto K$ is a strongly Gorenstein projective ideal of R. From [5, Proposition 2.12], there exists an exact sequence of D-modules: $0 \longrightarrow I \longrightarrow P \longrightarrow I \longrightarrow 0$, where P is finitely generated projective D-module. Moreover, $0 \longrightarrow I \propto K \longrightarrow P \otimes_D R \longrightarrow I \propto K \longrightarrow 0$, is an exact sequence of R-modules. Then, it remain to prove that $\mathrm{Ext}_R(I \propto K, R) =$

0. From [6, Proposition 4.1.3], $\mathrm{Ext}_R(I \propto K, R) \cong \mathrm{Ext}_D(I, R)$ and since R is a flat D-module $\mathrm{Ext}_D(I, R) = 0$, [5, Propositon 2.12]. Then, $\mathrm{Ext}_R(I \propto K, R) = 0$ and $I \propto K$ is finitely generated ideal of R strongly Gorenstein projective, so projective. Hence, $I \cong I \propto K \otimes_R D$ is a projective ideal of D.
(2) $\Rightarrow$ (1). Let J be a finitely generated ideal of R strongly Gorenstein projective. It is known from the presentation of Corollary 2.5 that the finitely generated ideals of R have the forms $I \propto K$ where I is a finitely generated ideal of D or $0 \propto E'$, where E' is a finitely generated D-submodule of K, (without loss of generality we can assume that E' is a finitely generated ideal of D). From Lemma 2.7, $J = I \propto K$ where I is an ideal finitely generated of D. Then, $J = I \propto K = I \otimes_D R$ and from [19, Theorem 2.1], I is a finitely generated ideal of D strongly Gorenstein projective, and by hypothesis I is projective. Then $J = I \otimes_D R$ is a projective ideal of R. ■

3 Examples

In this section, we give some examples of rings satisfy the properties studied in section 2.

Recall, from [8], a ring R is called an (n, d)-ring if every R-module having a finite n-presentation has projective dimension at most d. Also from [17], a commutative $(n, 0)-$ring R is called an n-Von Neumann regular ring. Thus , the 1-Von Neumann regular rings are the well-know Von Neumann regular rings. In the following result we show that the class of (n, d)-rings satisfy the property "all finitely generated strongly Gorenstein projective modules are projective".

Theorem 3.1
1. *Let R be an (n, d)-ring. Then, R-satisfies all finitely generated strongly Gorenstein projective R-modules are projective.*
2. *Let R be an n-Von Neumann regular ring. Then, there is not finitely generated ideal which is strongly Gorenstein projective.*

Proof.

1. Let M be a finitely generated strongly Gorenstein projective R-module. From [5, Proposition 2.12], there exists an exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0 \qquad (\star)$$

where P is finitely generated projective, then M is infinitely presented. Then from [17, Theorem 2.1], $\mathrm{pd}_R(M)$ is finite. Thus, from the exact sequence $(\star)$, we conclude that M is projective.

2. Suppose that I is a strongly Gorenstein projective ideal. From (2), I is projective, and since R is m-local I is free. Contradiction since any finitely generated ideal has a nonzero annihilator (by [17, Theorem 2.1]). ■

Next, we give examples of rings with weak global dimension infinite and which satisfies the property "all finitely generated strongly Gorenstein projective modules are projective".

Corollary 3.2 *Let K be a field and $A = K[[X_1, X_2, ...]]$ with $m = (X_1, X_2, ...)$ the maximal ideal of A. And let $R = A \propto (A/m)^\infty$. then:*

1. $\mathrm{wdim}(R) = \infty$.

2. *R satisfies all finitely generated strongly Gorenstein projective R-modules are projective.*
3. *There is not finitely generated ideal of R which is strongly Gorenstein projective.*

Proof. (1). Follows from [3, Theorem 3.1], since $(X_1, X_2, ...)$ is a minimal generating set of (m).
(2) and (3). Follows from Theorem 3.1 above and from [18, Theorem 2.1]. ■

Corollary 3.3 *Let (A, m) be an m-local ring such that m is finitely generated and $E = (A/m)^{\infty}$. Let $R = A \propto E$ the trivial ring extension of A by E. Then the following conditions holds in R :*

1. $\mathrm{wdim}(R) = \infty$.
2. *All finitely generated strongly Gorenstein projective modules are projective.*
3. *There is not finitely generated proper ideal which is strongly Gorenstein projective.*

Proof. (1). Follows from [3, Theorem 3.1].
(2) and (3). Use [18, Theorem 2.1] and Theorem 3.1 above. ■

Example 3.4 *Let K be a field and E a K-vector space of infinite dimension. Then, the property, all finitely generated strongly Gorenstein projective is projective, holds in $R = K \propto E$.*

Next, we see an example of ring which satisfies "all finitely generated strongly Gorenstein projective ideals are projective.

Example 3.5 *Let (D, m) an m-local integral domain and $K = qf(D)$. If $\mathrm{wdim}(D)$ is finite, then $D \propto K$ satisfies all finitely generated strongly Gorenstein projective ideals are projective.*

Proof. Follows from Theorem 2.1 and Theorem 2.6 above. ■

Next, we see an example of ring T with $\mathrm{G-gldim}(T) = 0$, over which the property "all (finitely generated) strongly Gorenstein projective modules are projective" does not holds. Also, this example prove that condition D is not a field in Theorem 2.6 is necessary.

Example 3.6 *Let K be a field and let $T = K \propto K$. Then $\mathrm{G-gldim}(T) = 0$ but T does not satisfies all (finitely generated) strongly Gorenstein projective modules are projective since $0 \propto K$ is a finitely generated ideal of T strongly Gorenstein projective but it is not projective.*

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